AAPT | UNITED STATES PHYSICS TEAM

AIP | 2005

**Solutions to Problems**

*Any correct solution should be awarded equivalent points. Suggested partial-credit points are presented in square brackets at the right margin. You may further break down the listed points into one point increments. Where alternate methods are presented, a student should be awarded points from only one method. Students should not be penalized in a subsequent part for using the wrong answer to a previous part. (No double jeopardy.) If it is clear they have done an intermediate step, they should get credit for it even if they have not presented it. For example, in 2.b., if a student wrote down $mvb = mv_f R$, they should get 10 points credit.*

Points

1. a. Setting the coordinate origin at the top of the cliff, the horizontal position is given by

$$x = v_{ix}t = (v_i\cos\theta)t = (v_i\cos 45.0^\circ)t = \frac{v_i t}{\sqrt{2}}. \qquad (1\text{-}1) \qquad [3]$$

Taking up to be positive, the vertical position is given by

$$y = v_{iy}t - \frac{1}{2}gt^2 = (v_i\sin 45.0^\circ)t - \frac{1}{2}gt^2 = \frac{v_i t}{\sqrt{2}} - \frac{1}{2}gt^2 \qquad (1\text{-}2) \qquad [4]$$

Solving (1-1) for $t$

$$t = \frac{x\sqrt{2}}{v_i} \qquad (1\text{-}3) \qquad [1]$$

and substituting into (1-2)

$$y = \frac{v_i}{\sqrt{2}}\left(\frac{x\sqrt{2}}{v_i}\right) - \frac{1}{2}g\left(\frac{x\sqrt{2}}{v_i}\right)^2 = x - g\frac{x^2}{v_i^2} \qquad [2]$$

Solving for $v_i$

$$v_i = \sqrt{\frac{gx^2}{x-y}} \qquad [1]$$

The shell lands at $x = 18.0$ m and $y = -4.00$ m

And

$$v_i = \sqrt{\frac{(9.8\ \text{m/s}^2)(18.0\ \text{m})^2}{18.0\ \text{m} - (-4.0\ \text{m})}} = 12.0\ \text{m/s} \qquad [2]$$

b. The y-component of the velocity is $v_y = v_{iy} - gt = v_i/\sqrt{2} - gt$ [2]

At its highest point $v_y = 0$ and $t_h = \dfrac{v_i}{g\sqrt{2}}$. (1-4) [2]

The shell's x position is

$$x = \frac{v_i t_h}{\sqrt{2}} = \frac{v_i}{\sqrt{2}}\left(\frac{v_i}{g\sqrt{2}}\right) = \frac{v_i^2}{2g} = \frac{(12.0\ \text{m/s})^2}{2(9.8\ \text{m/s}^2)} = 7.36\ \text{m} \qquad [1]$$

METHOD I: The explosion is an internal force and the motion of the shell's center of mass is unchanged. [2]

*(Note: Since fragment 1 has zero velocity after the explosion, the impulse is totally in the x-direction and the fragments land at the same time.)*

Let $x_1$ be the landing position of fragment $m_1$. Fragment 1 falls straight down, so

$x_1 = 7.36$ m while $x_2 = 30.0$ m [1]

The defining equation of the center of mass of the two fragments is

$$(m_1 + m_2)\, x_{cm} = m_1 x_1 + m_2 x_2 \,. \quad [2]$$

$$\frac{m_1}{m_2} = \frac{(x_2 - x_{cm})}{(x_{cm} - x_1)} = \frac{30.0 \text{ m-}18.0 \text{ m}}{18.0 \text{ m-}7.4 \text{ m}} = 1.13 \quad [2]$$

METHOD II: The explosion is an internal force and the total momentum of the shell is unchanged. [1]

*(Note: Since fragment 1 has zero velocity after the explosion, the impulse is totally in the x-direction and the y-component of momentum is unaffected.)*

For the x-component, momentum conservation gives

$$(m_1 + m_2) v_{ix} = m_1 (0) + m_2 v_2 = m_2 v_2 \quad [1]$$

The velocity of $m_2$ after the explosion is $v_2 = \dfrac{(m_1 + m_2) v_{ix}}{m_2} = \left(1 + \dfrac{m_1}{m_2}\right)\dfrac{v_i}{\sqrt{2}}$ [1]

The fragment's position after the explosion is $x_2 = x_0 + v_2 t_f = x_0 + \left(1 + \dfrac{m_1}{m_2}\right)\dfrac{v_i t_f}{\sqrt{2}}$, (1-5) [1]

where $t_f$ is the time it take the fragment to fall from its highest point to the ground. Since the y-component of velocity is unchanged by the explosion, the time to fall is the total time (1-3) minus the time to reach the highest point (1-4)

$$t_f = t - t_h = \frac{x\sqrt{2}}{v_i} - \frac{v_i}{g\sqrt{2}} = \frac{(18.0 \text{ m})\sqrt{2}}{(12.0 \text{ m/s})} - \frac{(12.0 \text{ m/s})}{(9.8 \text{ m/s}^2)\sqrt{2}} = 1.255 \text{ s} \quad [2]$$

*(The time can also be obtained by first finding the y-displacement at the highest point and then finding the time to fall from that point to the ground. Using either method a time of 1.26 s should get 2 points credit.)*

Solving (1-5) for the mass ratio

$$\frac{m_1}{m_2} = \frac{(x_2 - x_0)\sqrt{2}}{v_i t_f} - 1 = \frac{(30.0 \text{ m} - 7.36 \text{ m})\sqrt{2}}{(12.0 \text{ m/s})(1.255 \text{ s})} - 1 = 1.13 \quad [1]$$

2. a. Using energy conservation, the total mechanical energy when the probe is an infinite distance from the planet equals the total mechanical energy just before the probe hits.

$$K_\infty + U_\infty = K_p + U_p \quad [2]$$

Setting $U_\infty = 0$, [1]

we have $U_p = -\dfrac{GMm}{R}$ [2]

where m is the mass of the probe. The probe's kinetic energy is

$K_\infty = \frac{1}{2} m v^2$ and $K_p = \frac{1}{2} m v_p^{\,2}$ [2]

where $v_p$ is the probe's speed just before it hits the surface. Combining

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_p^2 - \frac{GMm}{R}. \qquad (2\text{-}1) \qquad [1]$$

Solving for $v_p$, we have $$v_p = \sqrt{v^2 + \frac{2GM}{R}}. \qquad [2]$$

b. A central force, such as the universal gravitational force, causes no torque. Therefore angular momentum is conserved. $L_\infty = L_p$ [2]

Angular momentum is given by $L = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$. (2-2) [2]

Conservation gives $\vec{r}_\infty \times m\vec{v} = \vec{r}_p \times m\vec{v}_p$ [1]

When the probe is very far away $|\vec{r}_\infty \times m\vec{v}| = mvr_\infty \sin\theta = mvb$. (2-3) [2]

When the probe has the largest speed it can have and still hit the planet, it will come in tangential to the surface, or perpendicular to the radius.

$$|\vec{r}_p \times m\vec{v}_p| = mv_p R \qquad (2\text{-}4) \qquad [2]$$

Combining (2-2), (2-3), and (2-4) $mvb = mv_p R$ [1]

or solving for $v_p$ $$v_p = v\frac{b}{R}. \qquad (2\text{-}5) \qquad [1]$$

Substituting (2-5) into (2-1) $$\frac{1}{2}mv^2 = \frac{1}{2}m\left(v\frac{b}{R}\right)^2 - \frac{GMm}{R} \qquad [2]$$

$$v^2\left[\left(\frac{b}{R}\right)^2 - 1\right] = \frac{2GM}{R}$$

$$v = \sqrt{\frac{2GM}{R\left[\left(\frac{b}{R}\right)^2 - 1\right]}} = \sqrt{\frac{2GMR}{b^2 - R^2}} \qquad [2]$$

3. In order for the mass to continue to oscillate back and forth, the spring force at maximum extension must exceed the maximum force of static friction. Letting $A_i$ equal the maximum displacement after the $i^{th}$ half cycle.

The mass stops oscillating when $F_s\big|_{x=A_i} < f_{s,\max}$ [2]

The spring force has magnitude $F_s = kx$ [1]

The maximum force of static friction is $f_{s,\max} = \mu_s N = \mu_s Mg$ [2]

The oscillation stops if $kA_i < \mu_s Mg$. [1]

Or numerically $$A_i < \frac{\mu_s Mg}{k} = \frac{(0.400)(10.00\text{ N})}{(100.0\text{ N/m})} = 0.0400\text{ m} \qquad (3\text{-}1) \qquad [1]$$

Applying the work energy theorem to a single half oscillation

$$W_{nc} + K_i + U_i = K_{i+1} + U_{i+1} \qquad (3\text{-}2) \qquad [2]$$

where $W_{nc}$ is the work done by non-conservative forces, friction in this case. At the extreme displacements $A_i$ and $A_{i+1}$, the kinetic energy is zero

at $x = A_i$ and $x = A_{i+1}$: $K_i = 0$ and $K_{i+1} = 0$. [1]

The spring potential energy at $A_i$ and $A_{i+1}$ is

$$U_i = \tfrac{1}{2}kA_i^2 \qquad \text{and} \qquad U_{i+1} = \tfrac{1}{2}kA_{i+1}^2. \qquad [1]$$

The work done by kinetic friction is $W_{nc} = -f_k d = -\mu_k Mgd$, [2]

where d is the total distance traveled, or $d = A_i + A_{i+1}$. [1]

$$W_{nc} = -\mu_k Mg(A_i + A_{i+1}) \qquad [1]$$

Substituting into (3-2) $-\mu_k Mg(A_i + A_{i+1}) + \tfrac{1}{2}kA_i^2 = \tfrac{1}{2}kA_{i+1}^2$. [1]

$$\mu_k Mg(A_i + A_{i+1}) = \tfrac{1}{2}k(A_i^2 - A_{i+1}^2) = \tfrac{1}{2}k(A_i + A_{i+1})(A_i - A_{i+1}).$$

Dividing by $(A_i + A_{i+1})$, $\mu_k Mg = \tfrac{1}{2}k(A_i - A_{i+1})$.

Solving for $A_{i+1}$ $A_{i+1} = A_i - \dfrac{2\mu_k Mg}{k}$ [2]

Numerically $A_{i+1} = A_i - \dfrac{2(0.200)(10.00\text{ N})}{(100.0\text{ N/m})} = A_i - 0.0400\text{ m}$.

Therefore each half cycle the amplitude decreases by 0.040 m = 4.0 cm. [1]

Successive amplitudes (starting with the initial displacement) are

0.180 m, 0.140 m, 0.100 m, 0.060 m, 0.020 m. [2]

At 0.020 m the amplitude satisfies condition (3-1) and the mass remains at rest. Since $A_0$ is to the left of the equilibrium position and they alternate, the stopping point is

0.020 m to the left of the equilibrium position. [2]

The total distance traveled is $D = A_0 + 2A_1 + 2A_2 + 2A_3 + A_4$

$$D = (0.180\text{ m}) + 2(0.140\text{ m}) + 2(0.100\text{ m}) + 2(0.060\text{ m}) + (0.020\text{ m}) = 0.800\text{ m} \qquad [2]$$

4. a. The translational kinetic energy is $K_t = \tfrac{1}{2}Mv^2$ or $K_t = W_F = Fd$ [2]

where $v$ is the time-dependent translational speed.

From Newton's Second Law with $F$ the only force acting, the acceleration of the center of mass is given by $a = \dfrac{F}{M}$. [1]

Starting from rest $v = at = \dfrac{Ft}{M}$ or $d = \tfrac{1}{2}at^2 = \dfrac{Ft^2}{2M}$ [2]

By either method, the translational kinetic energy is

$$K_t = \tfrac{1}{2}M\left(\frac{Ft}{M}\right)^2 = \frac{F^2t^2}{2M} \qquad \text{or} \qquad K_t = F\left(\frac{Ft^2}{2M}\right) = \frac{F^2t^2}{2M} \qquad (4\text{-}1) \qquad [1]$$

The rotational kinetic energy is $K_r = \tfrac{1}{2}I\omega^2$ or $K_r = \tau\theta$ [2]

where $\omega$ is the time-dependent rotational speed, $\theta$ is the angular displacement, $\tau$ is the torque, and $I$ the disk's rotational inertia about its center of mass. $I = \tfrac{1}{2}MR^2$. [1]

The torque about the center of mass is $|\vec{\tau}| = |\vec{R} \times \vec{F}| = RF$ [1]

From Newton's Second Law for rotational motion the angular acceleration about the center of mass is given by

$$\alpha = \frac{\tau}{I} = \frac{RF}{\tfrac{1}{2}MR^2} = \frac{2F}{MR}. \qquad (4\text{-}2) \qquad [1]$$

Starting from rest $\omega = \alpha t = \dfrac{2Ft}{MR}$ or $\theta = \frac{1}{2}\alpha t^2 = \dfrac{Ft^2}{MR}$ [2]

By either method, the rotational kinetic energy is

$$K_r = \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{2Ft}{MR}\right)^2 = \frac{F^2t^2}{M} \quad \text{or} \quad K_r = (RF)\frac{Ft^2}{MR} = \frac{F^2t^2}{M} \qquad (4\text{-}3) \qquad [1]$$

Dividing (4-3) by (4-1) $\dfrac{K_r}{K_t} = \dfrac{\left(F^2t^2/M\right)}{\left(F^2t^2/2M\right)} = 2$ [1]

*Note:* $\dfrac{K_r}{K_t} = \dfrac{\frac{1}{2}I\omega^2}{\frac{1}{2}Mv^2} = \dfrac{\left(\frac{1}{2}MR^2\right)\omega^2}{Mv^2} = \dfrac{1}{2}\left(\dfrac{\omega R}{v}\right)^2$ *This is not rolling without slipping and* $v \neq \omega R$.

*The answer ½ is only worth 4 points.*

b. The net work done by the force is equal to the change in kinetic energy.

$$W = \Delta K_t + \Delta K_r \qquad [2]$$

Since the disk starts at rest both initial kinetic energies are zero and

$$W = K_t + K_r = \frac{F^2t^2}{2M} + \frac{F^2t^2}{M} = \frac{3F^2t^2}{2M} \qquad (4\text{-}4) \qquad [3]$$

where $t$ is the time it takes to make the first rotation.

With uniform angular acceleration and starting from rest, $\theta = \frac{1}{2}\alpha t^2$. (4-5) [1]

From (4-2) $\alpha = \dfrac{2F}{MR}$ [1]

and for a single rotation $\theta = 2\pi$. [1]

Solving (4-5) for $t^2$ $t^2 = \dfrac{2\theta}{\alpha} = \dfrac{2(2\pi)}{(2F/MR)} = \dfrac{2\pi MR}{F}$ [1]

And substituting into (4-4) $W = \dfrac{3F^2t^2}{2M} = \dfrac{3F^2}{2M}\left(\dfrac{2\pi MR}{F}\right) = 3\pi FR$. [1]